Published by Institute of Physics Publishing for SISSA
Received: *February 28, 2006*
Revised: *May 8, 2006*
Accepted: *May 18, 2006*
Published: *June 12, 2006*

Liouville field theory with heavy charges, II. The conformal boundary case*

Pietro Menotti
Dipartimento di Fisica dell'Università
Pisa 56100, Italy, and
INFN, Sezione di Pisa
Italy
E-mail: menotti@df.unipi.it

Erik Tonni
Scuola Normale Superiore
Pisa 56100, Italy and
INFN, Sezione di Pisa
Italy
E-mail: e.tonni@sns.it

Abstract: We develop a general technique for computing functional integrals with fixed area and boundary length constraints. The correct quantum dimensions for the vertex functions are recovered by properly regularizing the Green function. Explicit computation is given for the one point function providing the first one loop check of the bootstrap formula.

Keywords: Conformal Field Models in String Theory, Field Theories in Lower Dimensions, Boundary Quantum Field Theory.

*This work is supported in part by M.I.U.R.

© SISSA 2006

JHEP06(2006)022

Contents

1. Introduction

In a preceding paper [1], denoted in the following by (I), we developed the heavy charge approach to the correlation functions in Liouville theory on the pseudosphere. Here we extend the treatment to the richer case of Liouville theory on a finite domain with conformally invariant boundary conditions. The bootstrap approach to such a problem was developed in the seminal papers by Fateev, Zamolodchikov and Zamolodchikov [2] and Teschner [3] providing several profound results; in particular the exact bulk one point function and the boundary two point function were derived. Further results were obtained in [4, 5]. As done in (I) for the pseudosphere, here we want to approach the problem in the standard way of quantum field theory, i.e. by computing first a stable classical background and then integrating over the quantum fluctuations.

In section 2 we separate the action into the classical and the quantum part and we derive the boundary conditions for the Green function.

In section 3 we develop the technique for computing the constrained path integrals by explicitly extracting the contribution of the fixed area and fixed boundary length constraints. Then we consider the transformation properties of the constrained N point vertex correlation functions under general conformal transformations. The key role of such development is played by the regularized value of the Green function at coincident points, both in the bulk and on the boundary. The non invariant regularization of the Green function

JHEP06(2006)022

suggested by Zamolodchikov and Zamolodchikov in the case of the pseudosphere [6 − 8] and its generalization to the boundary are essential. We prove that the one loop contribution (the quantum determinant) provides the correct quantum dimensions [10] to the vertex operators.

In section 4 we deal with the computation of the one point function. The background generated by a single charge is stable only in presence of a negative boundary cosmological constant; we compute the Green function on such a background satisfying the correct conformally invariant boundary conditions by explicitly resumming a Fourier series, as a more straightforward alternative to the general method employed in (I) for the pseudosphere. Such a Green function and its regularized value at coincident points are given in terms of the incomplete Beta function.

The presence of a negative boundary cosmological constant imposes to work with some constraints and the fixed boundary length constraint is the most natural one. It is proved that the fixed boundary length constraint is sufficient to make the functional integral well defined because the operator whose determinant provides the one loop contribution to the semiclassical result possesses one and only one negative eigenvalue. However, to compare our results with the ones given in [2] at fixed area A and fixed boundary length l, we introduce also the fixed area constraint. Exploiting the decomposition found in section 3, we are left with the computation of an unconstrained functional determinant, which we determine through the technique of varying the charges and the invariant ratio A/l^2.

The one loop result obtained in this way agrees with the expansion of the fixed area and boundary length one point function derived through the bootstrap method in [2] and for which there was up to now no perturbative check.

In the appendix we analyze the spectrum of the operator occurring in the quantum determinant.

2. Boundary Liouville field theory

The action on a finite simply connected domain Γ with background metric $g_{ab} = \delta_{ab}$ in absence of sources [2, 4] is

$$S_{\Gamma,0}[\phi] = \int_{\Gamma_\varepsilon} \left[\frac{1}{\pi} \partial_\zeta \phi \, \partial_{\bar\zeta} \phi + \mu \, e^{2b\phi} \right] d^2\zeta + \oint_{\partial\Gamma} \left[\frac{Q\,k}{2\pi} \phi + \mu_B \, e^{b\phi} \right] d\lambda \tag{2.1}$$

and in presence of sources it goes over to

$$\begin{aligned} S_{\Gamma,N}[\phi] = \lim_{\varepsilon_n \to 0} \Bigg\{ & \int_{\Gamma_\varepsilon} \left[\frac{1}{\pi} \partial_\zeta \phi \, \partial_{\bar\zeta} \phi + \mu \, e^{2b\phi} \right] d^2\zeta + \oint_{\partial\Gamma} \left[\frac{Q\,k}{2\pi} \phi + \mu_B \, e^{b\phi} \right] d\lambda \\ & - \frac{1}{2\pi i} \sum_{n=1}^{N} \alpha_n \oint_{\partial\gamma_n} \phi \left(\frac{d\zeta}{\zeta - \zeta_n} - \frac{d\bar\zeta}{\bar\zeta - \bar\zeta_n} \right) - \sum_{n=1}^{N} \alpha_n^2 \log \varepsilon_n^2 \Bigg\} \end{aligned} \tag{2.2}$$

where $Q = 1/b + b$, k is the extrinsic curvature of the boundary $\partial\Gamma$, defined as

$$k = \frac{1}{2i} \frac{d}{d\lambda} \left(\log \frac{d\zeta}{d\lambda} - \log \frac{d\bar\zeta}{d\lambda} \right), \qquad \zeta(\lambda) \in \partial\Gamma \tag{2.3}$$

JHEP06(2006)022

where λ is the parametric boundary length, i.e. $d\lambda = \sqrt{d\zeta d\bar{\zeta}}$. The integration domain $\Gamma_\varepsilon = \Gamma \backslash \bigcup_{n=1}^{N} \gamma_n$ is obtained by removing N infinitesimal disks $\gamma_n = \{|\zeta - \zeta_n| < \varepsilon_n\}$ from the simply connected domain Γ.

The boundary behavior of ϕ near the sources is

$$\phi(\zeta) \;=\; -\,\alpha_n \log|\zeta - \zeta_n|^2 \,+ O(1) \qquad \text{when} \qquad \zeta \,\to \zeta_n \,. \tag{2.4}$$

In order to connect the quantum theory to its semiclassical limit it is useful to define [11]

$$\varphi \;=\; 2b\phi\,, \qquad \alpha_n \;=\; \frac{\eta_n}{b}\,. \tag{2.5}$$

Then, we decompose the field φ as the sum of a classical background field φ_B and a quantum field

$$\varphi \;=\; \varphi_B + 2b\,\chi\,. \tag{2.6}$$

The condition of local finiteness of the area around each source and the asymptotic behavior (2.4) for the field ϕ imposes that $1 - 2\eta_n > 0$ [12–14].

Then, we can write the action as the sum of a classical and a quantum action as follows

$$S_{\Gamma,\,N}[\,\phi\,] \;=\; S_{cl}[\,\varphi_B\,] + S_q[\,\varphi_B,\,\chi\,]\,. \tag{2.7}$$

The classical action in absence of sources is given by

$$S_{cl,\,0}[\,\varphi_B\,] \;=\; \frac{1}{b^2}\left\{\int_\Gamma \left[\frac{1}{4\pi}\,\partial_\zeta\varphi_B\,\partial_{\bar\zeta}\varphi_B + \mu b^2\,e^{\varphi_B}\right] d^2\zeta \;+\; \oint_{\partial\Gamma}\left[\frac{k}{4\pi}\,\varphi_B + \mu_B b^2\,e^{\varphi_B/2}\right] d\lambda\right\} \tag{2.8}$$

and in presence of sources it goes over to

$$\begin{aligned} S_{cl}[\,\varphi_B\,] \;=\; \frac{1}{b^2}\,\lim_{\varepsilon_n\to 0}\Bigg\{ &\int_{\Gamma_\varepsilon}\left[\frac{1}{4\pi}\,\partial_\zeta\varphi_B\,\partial_{\bar\zeta}\varphi_B + \mu b^2\,e^{\varphi_B}\right] d^2\zeta + \oint_{\partial\Gamma}\left[\frac{k}{4\pi}\,\varphi_B + \mu_B b^2\,e^{\varphi_B/2}\right] d\lambda \\ &- \frac{1}{4\pi i}\,\sum_{n=1}^{N}\eta_n\oint_{\partial\gamma_n}\varphi_B\left(\frac{d\zeta}{\zeta-\zeta_n} - \frac{d\bar\zeta}{\bar\zeta-\bar\zeta_n}\right) - \sum_{n=1}^{N}\eta_n^2\log\varepsilon_n^2\Bigg\} \end{aligned} \tag{2.9}$$

while the quantum action reads

$$\begin{aligned} S_q[\,\varphi_B\,,\chi\,] = \lim_{\varepsilon_n\to 0}\Bigg\{&\int_{\Gamma_\varepsilon}\left[\frac{1}{\pi}\,\partial_\zeta\chi\,\partial_{\bar\zeta}\chi + \mu\,e^{\,\varphi_B}\left(e^{2b\,\chi}-1\right) - \frac{1}{\pi b}\,\chi\,\partial_\zeta\partial_{\bar\zeta}\varphi_B\right] d^2\zeta \\ &+ \frac{1}{4\pi i\,b}\oint_{\partial\Gamma}\chi\left(\partial_\zeta\varphi_B\,d\zeta - \partial_{\bar\zeta}\varphi_B\,d\bar\zeta\right) + \oint_{\partial\Gamma}\left[\frac{Q\,k}{2\pi}\,\chi + \mu_B\,e^{\varphi_B/2}\left(e^{b\,\chi}-1\right)\right] d\lambda \\ &+ \frac{1}{4\pi}\oint_{\partial\Gamma} k\;\varphi_B\,d\lambda - \frac{1}{2\pi i\,b}\sum_{n=1}^{N}\oint_{\partial\gamma_n}\chi\left(\frac{\eta_n}{\zeta-\zeta_n} + \frac{1}{2}\,\partial_\zeta\varphi_B\right) d\zeta \\ &+ \frac{1}{2\pi i\,b}\sum_{n=1}^{N}\oint_{\partial\gamma_n}\chi\left(\frac{\eta_n}{\bar\zeta-\bar\zeta_n} + \frac{1}{2}\,\partial_{\bar\zeta}\varphi_B\right) d\bar\zeta\Bigg\}\,. \end{aligned} \tag{2.10}$$

For the classical background field, we assume the following boundary behavior

$$\varphi_B(\zeta) = -2\eta_n \log|\zeta-\zeta_n|^2 + O(1) \qquad \text{when} \qquad \zeta \to \zeta_n\,. \tag{2.11}$$

Under a generic conformal transformation $\zeta \to \tilde\zeta = \tilde\zeta(\zeta)$ the background field changes as follows

$$\varphi_B(\zeta) \quad \longrightarrow \quad \tilde\varphi_B(\tilde\zeta) = \varphi_B(\zeta) - \log\left|\frac{d\tilde\zeta}{d\zeta}\right|^2 \tag{2.12}$$

so that $e^{\varphi_B}d^2\zeta$ is invariant, while the extrinsic curvature becomes

$$k \quad \longrightarrow \quad \tilde k = \frac{1}{\sqrt{J}}\left(k + \frac{1}{2i}\left(\frac{d\zeta}{d\lambda}\partial_\zeta \log J - \frac{d\bar\zeta}{d\lambda}\partial_{\bar\zeta}\log J\right)\right), \qquad \zeta(\lambda)\in\partial\Gamma \tag{2.13}$$

where $J \equiv |d\tilde\zeta/d\zeta|^2$. Under such conformal transformations the classical action both in absence and in presence of sources is invariant up to a field independent term. Thus, the classical action (2.9) by variation of the field φ_B gives rise to the conformally invariant field equation

$$-\partial_\zeta\partial_{\bar\zeta}\varphi_B + 2\pi\mu b^2 e^{\varphi_B} = 2\pi\sum_{n=1}^{N}\eta_n\,\delta^2(\zeta-\zeta_n) \tag{2.14}$$

which is the Liouville equation in presence of N sources, and to the following conformally invariant boundary conditions for the classical field

$$-\frac{1}{4\pi i}\left(\frac{d\zeta}{d\lambda}\partial_\zeta\varphi_B - \frac{d\bar\zeta}{d\lambda}\partial_{\bar\zeta}\varphi_B\right) = \frac{k}{2\pi} + \mu_B b^2 e^{\varphi_B/2}, \qquad \zeta(\lambda)\in\partial\Gamma\,. \tag{2.15}$$

The field independent terms which appear in the change of the actions under a conformal transformation are

$$\widetilde S_{cl,0}[\tilde\varphi_B] = S_{cl,0}[\varphi_B] + \frac{1}{8\pi b^2}\left(\oint_{\partial\tilde\Gamma}\tilde k\log\tilde J\,d\tilde\lambda - \oint_{\partial\Gamma}k\log J\,d\lambda\right) \tag{2.16}$$

where $\tilde J = |d\zeta/d\tilde\zeta|^2 = 1/J$, while in presence of sources we have

$$\begin{aligned}\widetilde S_{cl}[\tilde\varphi_B] = S_{cl}[\varphi_B] &+ \sum_{n=1}^{N}\frac{\eta_n(1-\eta_n)}{b^2}\log\left|\frac{d\tilde\zeta}{d\zeta}\right|^2_{\zeta=\zeta_n} \\ &+ \frac{1}{8\pi b^2}\left(\oint_{\partial\tilde\Gamma}\tilde k\log\tilde J\,d\tilde\lambda - \oint_{\partial\Gamma}k\log J\,d\lambda\right).\end{aligned} \tag{2.17}$$

The requirement that the expectation value of 1 be invariant under conformal transformations, i.e. the invariance of the vacuum, imposes to subtract the term

$$\frac{1}{8\pi b^2}\left(\oint_{\partial\tilde\Gamma}\tilde k\log\tilde J\,d\tilde\lambda - \oint_{\partial\Gamma}k\log J\,d\lambda\right) \tag{2.18}$$

from the r.h.s. of (2.16) and (2.17) when computing the transformation of the vertex correlation functions under conformal transformations. The term (2.18) vanishes identically for the conformal transformations which map the unit disk into itself, i.e. the $SU(1,1)$ transformations. In this way one obtains the semiclassical conformal dimensions of the

JHEP06(2006)022

vertex operators $e^{2(\eta_n/b)\phi(\zeta_n)}$

$$\frac{\eta_n(1-\eta_n)}{b^2} = \alpha_n\left(\frac{1}{b}-\alpha_n\right) . \tag{2.19}$$

Finally, we recall that μb^2 and $\mu_B b^2$ have to be kept constant when $b \to 0$ [2, 9].

Using the equation of motion for the classical field, the boundary conditions (2.15) and the behavior at the sources (2.11), the quantum action (2.10) becomes

$$\begin{aligned} S_q[\varphi_B\,,\chi] = & \int_\Gamma \left[\frac{1}{\pi}\,\partial_\zeta\chi\,\partial_{\bar\zeta}\chi + \mu\,e^{\varphi_B}\left(e^{2b\,\chi}-1-2b\chi\right)\right] d^2\zeta \\ & + \frac{1}{4\pi}\oint_{\partial\Gamma} k\,\varphi_B\,d\lambda + \frac{b}{2\pi}\oint_{\partial\Gamma} k\,\chi\,d\lambda + \oint_{\partial\Gamma}\mu_B\,e^{\varphi_B/2}\left(e^{b\,\chi}-1-b\chi\right)d\lambda\,. \end{aligned} \tag{2.20}$$

Integrating by parts the volume integral in (2.20) we obtain

$$\begin{aligned} S_q[\varphi_B\,,\chi] = & \int_\Gamma \left[-\frac{1}{\pi}\,\chi\,\partial_\zeta\partial_{\bar\zeta}\,\chi + \mu\,e^{\varphi_B}\left(e^{2b\,\chi}-1-2b\chi\right)\right] d^2\zeta \\ & + \frac{1}{4\pi i}\oint_{\partial\Gamma}\chi\left(\frac{d\zeta}{d\lambda}\,\partial_\zeta\chi - \frac{d\bar\zeta}{d\lambda}\,\partial_{\bar\zeta}\chi\right) \\ & + \frac{1}{4\pi}\oint_{\partial\Gamma} k\,\varphi_B\,d\lambda + \frac{b}{2\pi}\oint_{\partial\Gamma} k\,\chi\,d\lambda + \oint_{\partial\Gamma}\mu_B\,e^{\varphi_B/2}\left(e^{b\,\chi}-1-b\chi\right)d\lambda\,. \end{aligned} \tag{2.21}$$

By expanding in b the boundary conditions for the full field $\varphi = \varphi_B + 2b\chi$, which are

$$-\frac{1}{4\pi i}\left(\frac{d\zeta}{d\lambda}\,\partial_\zeta\varphi - \frac{d\bar\zeta}{d\lambda}\,\partial_{\bar\zeta}\varphi\right) = \frac{Q\,k}{2\pi}\,b + \mu_B b^2\,e^{\varphi/2}\,, \qquad \zeta(\lambda)\in\partial\Gamma \tag{2.22}$$

and using the boundary conditions (2.15) for the classical background field φ_B extracted from the classical action (2.9), we get the boundary conditions for χ

$$\begin{aligned} -\frac{1}{2\pi i}\left(\frac{d\zeta}{d\lambda}\,\partial_\zeta\chi - \frac{d\bar\zeta}{d\lambda}\,\partial_{\bar\zeta}\chi\right) & = \mu_B b\,e^{\varphi_B/2}\left(e^{b\chi}-1\right) + \frac{k}{2\pi}\,b\,, \qquad \zeta(\lambda)\in\partial\Gamma \\ & = \mu_B b^2\,e^{\varphi_B/2}\,\chi + b\left(\frac{k}{2\pi} + \mu_B b^2\,e^{\varphi_B/2}\,\frac{\chi^2}{2}\right) + O(b^2)\,. \end{aligned} \tag{2.23}$$

To order $O(b^0)$ we have

$$-\frac{1}{2\pi i}\left(\frac{d\zeta}{d\lambda}\,\partial_\zeta\chi - \frac{d\bar\zeta}{d\lambda}\,\partial_{\bar\zeta}\chi\right) = \mu_B b^2\,e^{\varphi_B/2}\,\chi\,, \qquad \zeta(\lambda)\in\partial\Gamma\,. \tag{2.24}$$

With the field χ satisfying (2.24), we are left with the following quantum action

$$\begin{aligned} S_q[\varphi_B\,,\chi] = & \frac{1}{2}\int_\Gamma \chi\left(-\frac{2}{\pi}\,\partial_\zeta\partial_{\bar\zeta} + 4\,\mu b^2\,e^{\varphi_B}\right)\chi\,d^2\zeta + \sum_{k\geqslant 3}\int_\Gamma \mu\,e^{\varphi_B}\,\frac{(2b\chi)^k}{k!}\,d^2z \\ & + \frac{1}{4\pi}\oint_{\partial\Gamma} k\,\varphi_B\,d\lambda + \frac{b}{2\pi}\oint_{\partial\Gamma} k\,\chi\,d\lambda + \sum_{k\geqslant 3}\oint_{\partial\Gamma}\mu_B\,e^{\varphi_B/2}\,\frac{(b\chi)^k}{k!}\,d\lambda\,. \end{aligned} \tag{2.25}$$

The first term of the second line is $O(b^0)$ while the other boundary terms are $O(b)$ or higher order in b.

JHEP06(2006)022

Thus, imposing on the Green function $g(\zeta,\zeta')$ of the following operator

$$D \equiv -\frac{2}{\pi}\,\partial_\zeta\partial_{\bar\zeta} + 4\,\mu b^2\, e^{\varphi_B} \tag{2.26}$$

the boundary conditions (2.24), i.e.

$$-\frac{1}{2\pi i}\left(\frac{d\zeta}{d\lambda}\,\partial_\zeta\, g(\zeta,\zeta') - \frac{d\bar\zeta}{d\lambda}\,\partial_{\bar\zeta}\, g(\zeta,\zeta')\right) = \mu_B b^2\, e^{\varphi_B/2}\, g(\zeta,\zeta')\,, \qquad \zeta(\lambda)\in\partial\Gamma \tag{2.27}$$

we can develop a perturbative expansion in b. The Green function of the operator D satisfies

$$D\, g(\zeta,\zeta') = \delta^2(\zeta-\zeta') \tag{2.28}$$

and, due to the covariance of D and of the boundary conditions (2.27), it is invariant in value under a conformal transformation $\zeta \to \tilde\zeta = \tilde\zeta(\zeta)$, i.e.

$$\tilde g(\tilde\zeta,\tilde\zeta') = g(\zeta,\zeta')\,. \tag{2.29}$$

3. Constrained path integral and quantum dimensions

The partition function in presence of sources is given by

$$Z(\zeta_1,\eta_1,\ldots,\zeta_N,\eta_N;\mu,\mu_B) = \int \mathcal{D}[\,\phi\,]\; e^{-S_{\Gamma,N}[\,\phi\,]} \tag{3.1}$$

with

$$Z(\zeta_1,\eta_1,\ldots,\zeta_N,\eta_N;\mu,\mu_B) \equiv \int_0^\infty \frac{dl}{l}\, e^{-\mu_B l} \int_0^\infty \frac{dA}{A}\, e^{-\mu A}\, Z(\zeta_1,\eta_1,\ldots,\zeta_N,\eta_N;A,l\,) \tag{3.2}$$

where we have used the conventions of [2] and

$$\begin{aligned} Z(\zeta_1,\eta_1,\ldots,\zeta_N,\eta_N;A,l\,) = e^{-S_{cl}[\varphi_B]}\, A\,l \int \mathcal{D}[\,\chi\,]\; e^{-S_q[\,\chi,\varphi_B\,]} \;\times \\ \times\, \delta\left(\int_\Gamma e^{\varphi_B+2b\chi} d^2\zeta - A\right) \delta\left(\oint_{\partial\Gamma} e^{\varphi_B/2+b\chi} d\lambda - l\right). \end{aligned} \tag{3.3}$$

The classical background field φ_B satisfies the Liouville equation (2.14) with boundary conditions (2.15) and

$$A = \int_\Gamma e^{\varphi_B} d^2\zeta \tag{3.4}$$

$$l = \oint_{\partial\Gamma} e^{\varphi_B/2}\, d\lambda\,. \tag{3.5}$$

Substituting (2.25) in (3.3) and exploiting (3.4) and (3.5), we have to one loop

$$Z(\zeta_1,\eta_1,\ldots,\zeta_N,\eta_N;A,l\,) = e^{-S_{cl}[\varphi_B]}\,\frac{A\,l}{2b^2}\, I \tag{3.6}$$

JHEP06(2006)022

where

$$I \;\equiv\; e^{-\frac{1}{4\pi}\oint_{\partial\Gamma} k\,\varphi_B\,d\lambda}\int\mathcal{D}[\,\chi\,]\;e^{-\frac{1}{2}(\chi,D\chi)}\;\delta\left(\int_\Gamma e^{\varphi_B}\chi\,d^2\zeta\right)\delta\left(\oint_{\partial\Gamma} e^{\varphi_B/2}\chi\,d\lambda\right)\;. \tag{3.7}$$

The seemingly non perturbative factor $1/b^2$ in (3.6) is due to the presence of the constraints.

Using the integral representation for the two delta functions [15] we have

$$\begin{aligned} I &= e^{-\frac{1}{4\pi}\oint_{\partial\Gamma} k\,\varphi_B\,d\lambda}\;\times \\ &\times\frac{1}{(2\pi)^2}\int\mathcal{D}[\,\chi\,]\int d\rho\int d\tau\;\exp\left\{-\frac{1}{2}\,(\chi,D\chi)+i\,\rho\int_\Gamma e^{\varphi_B}\chi\,d^2\zeta+i\,\tau\oint_{\partial\Gamma} e^{\varphi_B/2}\chi\,d\lambda\right\}. \end{aligned} \tag{3.8}$$

In the following we shall use the notation $\varphi_B(\lambda)$ to denote the field φ_B computed at the boundary point $\zeta(\lambda)\in\partial\Gamma$ and $g(\zeta,\lambda)$ and $g(\lambda,\lambda')$ to denote the values of the Green function with one or two arguments on the boundary.

Performing the field translation

$$\chi(\zeta)\;=\;\chi'(\zeta)+i\,\rho\int_\Gamma g(\zeta,\zeta')\,e^{\varphi_B(\zeta')}d^2\zeta'+i\,\tau\oint_{\partial\Gamma} g(\zeta,\lambda)\,e^{\varphi_B(\lambda)/2}d\lambda \tag{3.9}$$

we reach the result

$$I\;=\;\frac{e^{-\frac{1}{4\pi}\oint_{\partial\Gamma} k\,\varphi_B\,d\lambda}}{2\pi\sqrt{\det M\;\mathrm{Det}D}} \tag{3.10}$$

where M is the matrix

$$M\;=\;\begin{pmatrix} L & R\\ R & S\end{pmatrix} \tag{3.11}$$

with

$$L\;=\;\oint_{\partial\Gamma}\oint_{\partial\Gamma} e^{\varphi_B(\lambda)/2}d\lambda\;g(\lambda,\lambda')\,d\lambda'\,e^{\varphi_B(\lambda')/2} \tag{3.12}$$

$$S\;=\;\int_\Gamma\int_\Gamma e^{\varphi_B(\zeta)}d^2\zeta\;g(\zeta,\zeta')\,d^2\zeta'\,e^{\varphi_B(\zeta')} \tag{3.13}$$

$$R\;=\;\int_\Gamma\oint_{\partial\Gamma} e^{\varphi_B(\zeta)}d^2\zeta\;g(\zeta,\lambda)\,d\lambda\,e^{\varphi_B(\lambda)/2} \tag{3.14}$$

and $(\mathrm{Det}D)^{-1/2}$ is the unconstrained path integral

$$\left(\mathrm{Det}D\right)^{-1/2}\;=\;\int\mathcal{D}[\,\chi\,]\,e^{-\frac{1}{2}(\chi,D\chi)} \tag{3.15}$$

with χ satisfying the boundary conditions (2.24).

In section (4.4) it will be proved that the expression (3.10) holds also when the operator D has a finite number of negative eigenvalues, in which case $|\mathrm{Det}D|^{-1/2}$ is defined by

$$\prod_k\frac{\sqrt{2\pi}}{\sqrt{-\mu_k}}\int\mathcal{D}[\,\chi_\perp\,]\,e^{-\frac{1}{2}(\chi_\perp,D\chi_\perp)} \tag{3.16}$$

with k running over the negative eigenvalues μ_k and $\chi_\perp$ spans the subspace orthogonal to the eigenfunctions of D relative to the negative eigenvalues.

JHEP06(2006)022

We are interested in the transformation law of $I = I(\zeta_1, \eta_1, \ldots, \zeta_N, \eta_N; A, l\,)$ under a conformal transformation $\zeta \to \tilde{\zeta} = \tilde{\zeta}(\zeta)$.

We notice that the matrix elements of M are invariant under conformal transformations; hence we have to study the transformation properties of

$$I_1 \;\equiv\; e^{-\frac{1}{4\pi}\oint_{\partial\Gamma} k\,\varphi_B\,d\lambda} \int \mathcal{D}\,[\,\chi\,]\; e^{-\frac{1}{2}(\chi, D\chi)}\;. \tag{3.17}$$

To this end, we consider the eigenvalue equation

$$\left(-\frac{2}{\pi}\,\partial_\zeta\partial_{\bar\zeta} + 4\,\mu b^2\,e^{\varphi_B}\right)\chi_n \;=\; \mu_n\,\chi_n \tag{3.18}$$

with boundary conditions

$$-\,\frac{1}{2\pi i}\,\left(\frac{d\zeta}{d\lambda}\,\partial_\zeta\chi_n \;-\; \frac{d\bar\zeta}{d\lambda}\,\partial_{\bar\zeta}\chi_n\right) \;=\; \mu_B b^2\,e^{\varphi_B/2}\,\chi_n\,, \qquad \zeta(\lambda)\in\partial\Gamma\,. \tag{3.19}$$

Taking the variation of (3.18), we get

$$\left(-\frac{2}{\pi}\,\partial_z\partial_{\bar z} + 4\,\mu b^2\,e^{\varphi_B}\right)\delta\chi_n + \,4\,\chi_n\,\delta(\mu b^2 e^{\varphi_B}\,) \;=\; \delta\mu_n\,\chi_n + \mu_n\,\delta\chi_n\;. \tag{3.20}$$

Then we multiply (3.20) by χ_n and we integrate the result on the domain Γ. Exploiting the orthonormality of the eigenfunctions χ_n, the eigenvalue equation (3.18) and the divergence theorem, we get

$$\delta\mu_n \;=\; 4\int_\Gamma \chi_n^2\,\delta\big(\mu b^2 e^{\varphi_B}\big)\,d^2\zeta \;-\; \frac{1}{2\pi}\oint_{\partial\Gamma}\big(\,\chi_n\,\partial_{\hat n}\delta\chi_n -\, \delta\chi_n\,\partial_{\hat n}\chi_n\,\big)d\lambda \tag{3.21}$$

where $\partial_{\hat n}$ denotes the outward normal derivative on the boundary

$$\partial_{\hat n} \;=\; \frac{1}{i}\left(\frac{d\zeta}{d\lambda}\,\partial_\zeta \;-\; \frac{d\bar\zeta}{d\lambda}\,\partial_{\bar\zeta}\right)\,, \qquad \zeta(\lambda)\in\partial\Gamma\,. \tag{3.22}$$

On the other hand, the variation of the boundary conditions (3.19) gives

$$-\,\frac{1}{2\pi i}\,\left(\frac{d\zeta}{d\lambda}\,\partial_\zeta\delta\chi_n \;-\; \frac{d\bar\zeta}{d\lambda}\,\partial_{\bar\zeta}\delta\chi_n\right) \;=\; \delta\big(\mu_B b^2 e^{\varphi_B/2}\big)\,\chi_n + \mu_B b^2 e^{\varphi_B/2}\,\delta\chi_n\,, \qquad \zeta(\lambda)\in\partial\Gamma\;. \tag{3.23}$$

Using (3.19) and (3.23), we find that (3.21) becomes

$$\delta\mu_n \;=\; 4\int_\Gamma \chi_n^2(\zeta)\,\delta\big(\mu b^2 e^{\varphi_B}\big)\,d^2\zeta \;+\; \oint_{\partial\Gamma}\chi_n^2(\lambda)\,\delta\big(\mu_B b^2 e^{\varphi_B/2}\big)\,d\lambda\;. \tag{3.24}$$

At this point, exploiting the spectral representation of the Green function, i.e.

$$g(\zeta,\zeta') \;=\; \sum_{n\geqslant 1}\frac{\chi_n(\zeta)\chi_n(\zeta')}{\mu_n} \tag{3.25}$$

JHEP06(2006)022

we get the variation

$$\delta\left(\log\left(\operatorname{Det} D\right)^{-1/2}\right) = -\frac{1}{2}\sum_{n\geqslant 1}\frac{\delta\mu_n}{\mu_n}$$
$$= -2\int_\Gamma g(\zeta,\zeta)\,\delta\big(\mu b^2 e^{\varphi_B}\big)\,d^2\zeta\;-\;\frac{1}{2}\oint_{\partial\Gamma} g_B(\lambda,\lambda)\,\delta\big(\mu_B b^2 e^{\varphi_B/2}\big)\,d\lambda \tag{3.26}$$

where the Green function at coincident points in the bulk and on the boundary appear. Such quantities are divergent and have to be regularized.

We have already learnt that the correct regularization is the one suggested by Zamolodchikov and Zamolodchikov [6, 1], i.e.

$$g(\zeta,\zeta) \equiv \lim_{\zeta'\to\zeta}\left\{ g(\zeta,\zeta') + \frac{1}{2}\log|\,\zeta-\zeta'|^2\right\} \tag{3.27}$$

while $g_B(\lambda,\lambda)$ will be similarly defined by simply subtracting the logarithmic divergence.

Notice that $g_B(\lambda,\lambda')$ diverges like $\log|\lambda-\lambda'|^2$ when $\lambda'\to\lambda$ and not like $1/2\log|\lambda-\lambda'|^2$, as one could naively expect. A general argument for this behavior is the following.[1]

After having transformed the simply connected domain Γ into the upper half plane $\mathbb{H}$, the Green function $g_N(\xi,\xi')$ for the operator D with Neumann boundary conditions satisfies

$$\left(\frac{d}{d\xi}-\frac{d}{d\bar\xi}\right) g_N(\xi,\xi') = 0 \qquad \text{when} \qquad \xi\in\mathbb{R} \tag{3.28}$$

hence its behavior near the boundary ($\mathrm{Im}\xi\to 0$) is given by the method of the images, i.e.

$$g_N(\xi,\xi') = -\frac{1}{2}\log(\xi-\xi')(\bar\xi-\bar\xi') - \frac{1}{2}\log(\xi-\bar\xi')(\bar\xi-\xi') + \dots \tag{3.29}$$

which satisfies (3.28).

The complete Green function $g(\xi,\xi')$ with mixed boundary conditions (2.27) has the form

$$g(\xi,\xi') = A(\xi,\xi')\left(-\frac{1}{2}\log(\xi-\xi')(\bar\xi-\bar\xi') - \frac{1}{2}\log(\xi-\bar\xi')(\bar\xi-\xi') + C(\xi,\xi')\right) \tag{3.30}$$

where $A(\xi,\xi')$ and $C(\xi,\xi')$ are regular functions [16] with $A(\xi,\xi)=1$. The mixed boundary conditions (2.27) for $g(\xi,\xi')$ then read

$$\frac{g(\xi,\xi')}{A(\xi,\xi')}\left(\frac{d}{d\xi}-\frac{d}{d\bar\xi}\right)A(\xi,\xi') + A(\xi,\xi')\left(\frac{d}{d\xi}-\frac{d}{d\bar\xi}\right)C(\xi,\xi') = -2\pi i\,\mu_B b^2\,e^{\varphi_B/2}\,g(\xi,\xi') \tag{3.31}$$

for $\xi\in\mathbb{R}$, i.e.

$$\begin{cases} \left(\dfrac{d}{d\xi}-\dfrac{d}{d\bar\xi}\right)A(\xi,\xi') = -2\pi i\,\mu_B b^2\,e^{\varphi_B/2}A(\xi,\xi') & \text{when} \quad \xi\in\mathbb{R} \\ \left(\dfrac{d}{d\xi}-\dfrac{d}{d\bar\xi}\right)C(\xi,\xi') = 0 & \text{when} \quad \xi\in\mathbb{R}\,. \end{cases} \tag{3.32}$$

[1] We are grateful to Giovanni Morchio for providing the described argument.

JHEP06(2006)022

In the bulk for $\xi = \xi'$ we have

$$g(\xi, \xi') \simeq -\frac{1}{2} \log |\xi - \xi'|^2 - \frac{1}{2} \log |\xi - \bar{\xi}|^2 + C(\xi, \xi) \tag{3.33}$$

while for both $\xi = x$ and $\xi' = x'$ on the boundary $\mathbb{R} = \partial\mathbb{H}$ we have

$$g(x, x') \simeq - \log |x - x'|^2 + C(x, x) \,. \tag{3.34}$$

We notice that the finite part of $g(\xi, \xi')$ in the bulk for ξ going to the boundary coincides with the finite part of $g(x, x')$ on the boundary, which is given by $C(x, x)$. Such a boundary behavior will be verified explicitly for the Green function on the background generated by one source in section 4.2, where also the finite terms at coincident points will be computed.

Thus, coming back to the general simply connected domain Γ, we define the regularized value of the Green function on the boundary at coincident points as follows

$$g_B(\lambda, \lambda) \equiv \lim_{\lambda' \to \lambda} \left\{ g(\lambda, \lambda') + \log \left|\lambda - \lambda'\right|^2 \right\} \,. \tag{3.35}$$

Now we observe that, since the Green function $g(\zeta, \zeta')$ is invariant in value under a conformal transformation $\zeta \to \tilde{\zeta} = \tilde{\zeta}(\zeta)$, then its regularized values at coincident points change as follows

$$g(\zeta, \zeta) \longrightarrow \tilde{g}(\tilde{\zeta}, \tilde{\zeta}) = g(\zeta, \zeta) + \frac{1}{2} \log \left|\frac{d\tilde{\zeta}}{d\zeta}\right|^2 \qquad \text{when} \qquad \zeta \in \Gamma \tag{3.36}$$

and

$$g_B(\lambda, \lambda) \longrightarrow \tilde{g}_B(\tilde{\lambda}, \tilde{\lambda}) = g_B(\lambda, \lambda) + \log \left|\frac{d\tilde{\zeta}}{d\zeta}\right|^2, \qquad \zeta(\lambda) \in \partial\Gamma \,. \tag{3.37}$$

We shall compute the change of (3.17) $I_1 \to \tilde{I}_1$ under a conformal transformation by computing the transformation properties of its derivatives w.r.t. $\eta_1, \ldots, \eta_N$, A and l.

The logarithmic variation of $\tilde{I}_1$ is given by

$$\begin{aligned} \delta \log \tilde{I}_1 = \delta \left(-\frac{1}{4\pi} \oint_{\partial\tilde{\Gamma}} \tilde{k}\, \tilde{\varphi}_B \, d\tilde{\lambda} \right) \\ -2 \int_{\tilde{\Gamma}} \tilde{g}(\tilde{\zeta}, \tilde{\zeta})\, \delta\big(\mu b^2 e^{\tilde{\varphi}_B}\big)\, d^2\tilde{\zeta} - \frac{1}{2} \oint_{\partial\tilde{\Gamma}} \tilde{g}_B(\tilde{\lambda}, \tilde{\lambda})\, \delta\big(\mu_B b^2 e^{\tilde{\varphi}_B/2}\big)\, d\tilde{\lambda} \,. \end{aligned} \tag{3.38}$$

The terms in the second line can be rewritten as

$$\begin{aligned} &- 2 \int_{\tilde{\Gamma}} \tilde{g}(\tilde{\zeta}, \tilde{\zeta})\, \delta\big(\mu b^2 e^{\tilde{\varphi}_B}\big)\, d^2\tilde{\zeta} - \frac{1}{2} \oint_{\partial\tilde{\Gamma}} \tilde{g}_B(\tilde{\lambda}, \tilde{\lambda})\, \delta\big(\mu_B b^2 e^{\tilde{\varphi}_B/2}\big)\, d\tilde{\lambda} = \\ &= -2 \int_{\Gamma} g(\zeta, \zeta)\, \delta\big(\mu b^2 e^{\varphi_B}\big)\, d^2\zeta - \frac{1}{2} \oint_{\partial\Gamma} g_B(\lambda, \lambda)\, \delta\big(\mu_B b^2 e^{\varphi_B/2}\big)\, d\lambda \\ &\quad - \int_{\Gamma} \log J \, \delta\big(\mu b^2 e^{\varphi_B}\big)\, d^2\zeta - \frac{1}{2} \oint_{\partial\Gamma} \log J \, \delta\big(\mu_B b^2 e^{\varphi_B/2}\big)\, d\lambda \end{aligned} \tag{3.39}$$

where $J \equiv |d\tilde{\zeta}/d\zeta|^2$ is independent of $\eta_1, \ldots, \eta_N$, A and l.

JHEP06(2006)022

Using the Liouville equation (2.14) for φ_B and the boundary conditions (2.15), we obtain for last two terms in (3.39)

$$-\delta\left(\sum_{j=1}^{N}\eta_j\log J|_{\zeta_j}\right)+\delta\left[\frac{1}{8\pi i}\oint_{\partial\Gamma}\varphi_B\Big(\partial_\zeta\log J\,d\zeta\,-\,\partial_{\bar\zeta}\log J\,d\bar\zeta\Big)+\frac{1}{4\pi}\oint_{\partial\Gamma}k\,\log J\,d\lambda\right]. \tag{3.40}$$

The term in (3.38) containing the curvature $\tilde{k}$ becomes

$$\delta\left[-\frac{1}{4\pi}\oint_{\partial\Gamma}k\,\varphi_B\,d\lambda-\frac{1}{8\pi i}\oint_{\partial\Gamma}\varphi_B\Big(\partial_\zeta\log J\,d\zeta\,-\,\partial_{\bar\zeta}\log J\,d\bar\zeta\Big)-\frac{1}{4\pi}\oint_{\partial\tilde\Gamma}\tilde{k}\,\log\tilde{J}\,d\tilde\lambda\right] \tag{3.41}$$

where we have used the transformation law (2.13) for k under conformal transformations.

Summing the two contributions and taking into account that the term

$$\frac{1}{4\pi}\left(\oint_{\partial\tilde\Gamma}\tilde{k}\,\log\tilde{J}\,d\tilde\lambda\,-\,\oint_{\partial\Gamma}k\,\log J\,d\lambda\right) \tag{3.42}$$

does not depend on $\eta_1,\ldots,\eta_N$, A and l, we find that

$$\delta\,\log\tilde{I}_1\;=\;\delta\,\log I_1\,-\,\delta\left(\sum_{j=1}^{N}\eta_j\,\log J|_{\zeta_j}\right) \tag{3.43}$$

which gives

$$\log\tilde{I}_1\;=\;\log I_1\,-\,\sum_{j=1}^{N}\eta_j\,\log J|_{\zeta_j}\,+\,f(\zeta_1,\ldots,\zeta_N) \tag{3.44}$$

where $f(\zeta_1,\ldots,\zeta_N)$ is independent of $\eta_1,\ldots,\eta_N$, A and l. Since for vanishing η_1 the vertex correlation function has to be independent of ζ_1, we have that $f(\zeta_1,\ldots,\zeta_N)$ does not depend on ζ_1 and, similarly, on $\zeta_2,\ldots,\zeta_N$.

As the conformal dimensions Δ_{α_k} are given by

$$-\,\Delta_{\alpha_k}\frac{\partial\log J|_{\zeta_k}}{\partial\zeta_k}\;=\;\frac{\partial}{\partial\zeta_k}\,\log\frac{\langle\,e^{2\alpha_1\tilde\phi(\tilde\zeta_1)}\ldots e^{2\alpha_N\tilde\phi(\tilde\zeta_N)}\,\rangle}{\langle\,e^{2\alpha_1\phi(\zeta_1)}\ldots e^{2\alpha_N\phi(\zeta_N)}\,\rangle} \tag{3.45}$$

the relation (3.44) provides the one loop quantum correction to the semiclassical dimensions

$$\frac{\eta(1-\eta)}{b^2}\qquad\longrightarrow\qquad\Delta_{\eta/b}\;=\;\frac{\eta(1-\eta)}{b^2}\,+\,\eta\;=\;\alpha\left(\frac{1}{b}+b-\alpha\right) \tag{3.46}$$

which coincide with the exact quantum dimensions [10]. In particular the weights of the bulk cosmological term $e^{2b\phi}$ become $(1,1)$.

4. The one point function

Through a conformal transformation, one can always reduce the finite simply connected domain Γ to the unit disk Δ. The classical and the quantum actions are given by (2.9) and (2.25) respectively, with $k=1$. The parametric boundary length in the case of the unit disk Δ is given by the angular coordinate θ.

We shall consider the one point function, i.e. one single source of charge $\eta_1=\eta$ placed in $z_1=0$, without loss of generality.

JHEP06(2006)022

4.1 The classical action

The solution of the Liouville equation (2.14) with $N = 1$ on the unit disk is [2, 17]

$$e^{\varphi_c} = \frac{1}{\pi\mu b^2}\,\frac{a^2(1-2\eta)^2}{\big((z\bar z)^\eta - a^2 (z\bar z)^{1-\eta}\big)^2} \qquad \mu > 0\,, \quad 0 < a^2 < 1 \tag{4.1}$$

with $\mu > 0$ and $1 - 2\eta > 0$. The condition $a^2 < 1$ is necessary to avoid singularities inside Δ except for the one placed in 0. The boundary conditions (2.15) when $\Gamma = \Delta$ read

$$- r^2 \partial_{r^2} \varphi_c = 1 + 2\pi\,\mu_B b^2\, e^{\varphi_c/2} \qquad \text{when} \qquad r \equiv |z| = 1 \tag{4.2}$$

and this condition on the solution (4.1) provides the following relation between a^2 and the scale invariant ratio of the cosmological constants

$$\sqrt{\pi}\, b\, \frac{\mu_B}{\sqrt{\mu}} = -\,\frac{1+a^2}{2|a|}\,. \tag{4.3}$$

It is important to remark that the semiclassical limit can be realized only for $\mu_B < 0$. More precisely, from (4.3), we find that the scale invariant ratio of the cosmological constants has to be $\sqrt{\pi}\, b\, \mu_B/\sqrt{\mu} < -1$.

The classical field (4.1) gives rise to specific expressions for the area A and the boundary length l of the unit disk in terms of the bulk cosmological constant μ, the charge η and parameter a^2

$$A = \int_\Delta e^{\varphi_c} d^2 z = \frac{1}{\mu b^2}\,\frac{a^2(1-2\eta)}{1-a^2} \tag{4.4}$$

$$l = \oint_{\partial\Delta} e^{\varphi_c/2}\, d\theta = \frac{\sqrt{\pi}}{b\sqrt{\mu}}\,\frac{2|a|\,(1-2\eta)}{1-a^2} = -\,\frac{1}{\mu_B b^2}\,\frac{(1-2\eta)(1+a^2)}{1-a^2} \tag{4.5}$$

where in the last step of (4.5) we have employed (4.3). A useful relation we shall employ in the following is

$$a^2 = 1 - 4\pi\,\frac{A}{l^2}\,(1-2\eta)\,. \tag{4.6}$$

Given the classical solution (4.1), we can compute the classical action (2.9) on such a background. The result is

$$S_{cl}[\,\varphi_c\,] = \frac{S_0(\eta; A, l)}{b^2} + \mu\, A + \mu_B\, l \tag{4.7}$$

where [2]

$$\begin{aligned} S_0(\eta; A, l) &= b^2\, S_{cl}[\,\varphi_c\,]\Big|_{\mu=\mu_B=0} = \frac{l^2}{4\pi A} + (1-2\eta)\left(\log\frac{2A}{l} + \log(1-2\eta) - 1\right) \\ &= (1-2\eta)\left(\frac{1}{1-a^2} + \log|a| - \frac{1}{2}\log(\pi\mu b^2) + \log(1-2\eta) - 1\right). \end{aligned} \tag{4.8}$$

JHEP06(2006)022

4.2 The Green function

The Green function on the background generated by one heavy charge satisfies the following equation

$$\begin{aligned} D\, g(z,t) &= \left(-\frac{2}{\pi}\,\partial_z\partial_{\bar z} \,+\, 4\,\mu b^2\, e^{\varphi_c}\right) g(z,t) \\ &= \left(-\frac{2}{\pi}\,\partial_z\partial_{\bar z} \,+\, \frac{4\,a^2(1-2\eta)^2}{\pi\big((z\bar z)^\eta - a^2 (z\bar z)^{1-\eta}\big)^2}\right) g(z,t) \;=\; \delta^2(z-t) \end{aligned} \tag{4.9}$$

and its boundary conditions are

$$-\,r^2\frac{\partial}{\partial r^2}\, g(z,t) \;=\; \pi\,\mu_B b^2\, e^{\varphi_c/2}\, g(z,t) \qquad \text{when} \qquad r^2 = 1 \tag{4.10}$$

where $z = re^{i\theta}$ and φ_c is the classical background field (4.1). Exploiting the relation (4.3) derived from the boundary conditions of φ_c, the boundary conditions for the Green function read

$$\big(\, z\,\partial_z + \bar z\,\partial_{\bar z}\,\big)\, g(z,t) \;=\; (1-2\eta)\,\frac{1+a^2}{1-a^2}\, g(z,t) \qquad \text{when} \qquad |z| = 1\,. \tag{4.11}$$

To compute $g(z,t)$ in the simplest way, we expand it as a sum of partial waves

$$g(z,t) \;=\; \sum_{m\geqslant 0}\, g_m(x,y)\, \cos\big(m(\theta_x-\theta_y)\big) \tag{4.12}$$

where $x = |z|^2$ and $y = |t|^2$. The Fourier coefficients $g_m(x,y)$ are symmetric in the arguments and satisfy the following equation

$$\left(-\,2\,\frac{\partial}{\partial x}\left(x\,\frac{\partial}{\partial x}\right) + \frac{m^2}{2\,x} + \frac{4\,a^2(1-2\eta)^2}{(x^\eta - a^2 x^{1-\eta})^2}\right) g_m(x,y) \;=\; d_m\,\delta(x-y) \tag{4.13}$$

with $d_0 = 1$ and $d_m = 2$ for $m \geqslant 1$. They are given by

$$g_m(x,y) \;=\; \theta(y-x)\, a_m(x)\, b_m(y) + \theta(x-y)\, a_m(y)\, b_m(x) \qquad \forall\, m \geqslant 0 \tag{4.14}$$

where both $a_m(x)$ and $b_m(x)$ satisfy the homogenous version of (4.13). The functions $a_m(x)$ must be regular in $x = 0$ and, to reproduce the delta singularity, the wronskian of the solutions a_m and b_m must be

$$\begin{cases} \dfrac{\partial a_0(r^2)}{\partial r}\, b_0(r^2) \;-\; \dfrac{\partial b_0(r^2)}{\partial r}\, a_0(r^2) \;=\; \dfrac{1}{r} \\[2ex] \dfrac{\partial a_m(r^2)}{\partial r}\, b_m(r^2) \;-\; \dfrac{\partial b_m(r^2)}{\partial r}\, a_m(r^2) \;=\; \dfrac{2}{r} \qquad\qquad m \geqslant 1\,. \end{cases} \tag{4.15}$$

The boundary conditions (4.10) are translated into

$$2\,y\,\frac{\partial}{\partial y}\, b_m(y) \;=\; (1-2\eta)\,\frac{1+a^2}{1-a^2}\, b_m(y) \qquad \text{when} \qquad y \,=\, 1\,; \qquad \forall\, m \geqslant 0\,. \tag{4.16}$$

JHEP06(2006)022

The solutions for $m = 0$ are

$$a_0(x) = \frac{1+a^2x^{1-2\eta}}{1-a^2x^{1-2\eta}} \qquad b_0(y) = -\frac{1}{2(1-2\eta)}\left(\frac{1+a^2y^{1-2\eta}}{1-a^2y^{1-2\eta}}\log y^{1-2\eta}+2\right) \tag{4.17}$$

while $a_m(x)$ and $b_m(y)$ for $m \geqslant 1$ read

$$a_m(x) = \frac{x^{m/2}}{1-a^2x^{1-2\eta}}\left(1-\frac{m-(1-2\eta)}{m+(1-2\eta)}\,a^2x^{1-2\eta}\right) \tag{4.18}$$

$$b_m(y) = -\frac{y^{-m/2}}{m\big(m-(1-2\eta)\big)}\left((1-2\eta)\frac{1+a^2y^{1-2\eta}}{1-a^2y^{1-2\eta}}(1-y^m)-m(1+y^m)\right). \tag{4.19}$$

For $a^2 \to 1$, the expressions of $a_m(x)$ and $b_m(y)$ go over to their counterparts on the pseudosphere [1].

Given $a_m(x)$ and $b_m(y)$, the series (4.12) can be explicitly summed [1, 18]. The result is

$$\begin{aligned} g(z,t) = &-\frac{1}{2}\frac{1+a^2(z\bar z)^{1-2\eta}}{1-a^2(z\bar z)^{1-2\eta}}\left\{\frac{1+a^2(t\bar t)^{1-2\eta}}{1-a^2(t\bar t)^{1-2\eta}}\log\omega(z,t)+\frac{2}{1-2\eta}\right\} \\ &-\frac{1}{1-a^2(z\bar z)^{1-2\eta}}\,\frac{1}{1-a^2(t\bar t)^{1-2\eta}}\times \\ &\times\Bigg\{a^2\frac{(t\bar t)^{1-2\eta}}{2\eta}\frac{z}{t}F(2\eta,1;1+2\eta;z/t)+a^2\frac{(z\bar z)^{1-2\eta}}{2(1-\eta)}\frac{z}{t}F(2-2\eta,1;3-2\eta;z/t)+\text{c.c.} \\ &-\frac{1}{2\eta}z\bar tF(2\eta,1;1+2\eta;z\bar t)-a^4\frac{(z\bar z)^{1-2\eta}(t\bar t)^{1-2\eta}}{2(1-\eta)}z\bar tF(2-2\eta,1;3-2\eta;z\bar t)+\text{c.c.}\Bigg\}. \end{aligned} \tag{4.20}$$

This Green function can be also obtained by applying the general method developed in [1, 19, 20].

In the limit $a^2 \to 1$ for z and t fixed we recover the Green function of the pseudosphere [1, 19], which has also a well defined limit $\eta \to 0$. On the other hand the limit $\eta \to 0$ of $g(z,t)$ for fixed $a^2 < 1$ is singular and this fact is related to the occurrence of a zero mode when $\eta = 0$ (see the appendix). Thus the two limits $a^2 \to 1$ and $\eta \to 0$ of the Green function (4.20) do not commute.

The regularized value $g(z,z)$ of this Green function at coincident point is defined in (3.27). To compute it, we can expand $\log|z-t|^2$ as a Fourier series with symmetric and factorized coefficients by employing

$$\frac{1}{2}\log|z-t|^2 = \frac{1}{2}\log y - \sum_{m\geqslant 1}\frac{1}{m}\left(\frac{x}{y}\right)^{m/2}\cos\big(m(\theta_x-\theta_y)\big) \tag{4.21}$$

where $x = \min(|z|,|t|)$ and $y = \max(|z|,|t|)$. Adding (4.21) to (4.12) and computing the result at coincident points, we get a series representation for $g(z,z)$, which can be summed explicitly.

JHEP06(2006)022

Otherwise, we can apply directly the definition (3.27) to (4.20), obtaining the same result, i.e.

$$g(z,z)=\left(\frac{1+a^2(z\bar z)^{1-2\eta}}{1-a^2(z\bar z)^{1-2\eta}}\right)^2\log(1-z\bar z)\;-\;\frac{1}{1-2\eta}\,\frac{1+a^2(z\bar z)^{1-2\eta}}{1-a^2(z\bar z)^{1-2\eta}} \qquad (4.22)$$

$$+\,\frac{2\,(z\bar z)^{1-2\eta}}{\big(1-a^2(z\bar z)^{1-2\eta}\big)^2}\Big(B_{z\bar z}(2\eta\,,0)+a^4B_{z\bar z}(2-2\eta\,,0)$$

$$+a^2\big(\,2\gamma_{\scriptscriptstyle E}+\psi(2\eta)+\psi(2-2\eta)-\log z\bar z\,\big)\Big)\,.$$

where $\gamma_{\scriptscriptstyle E}$ is the Euler constant and $\psi(x)=\Gamma'(x)/\Gamma(x)$.

For $a^2\to 1$ $g(z,z)$ in the bulk becomes the corresponding function on the pseudo-sphere [1, 19], hence the two limits $a^2\to 1$ and $t\to z$ of the Green function (4.20) commute.

By using the expansion of the incomplete Beta function $B_x(\alpha,0)$ around $x=1$ [1, 18], we find that the boundary behavior of $g(z,z)$ is

$$g(z,z)\;=\;-\;\log(1-z\bar z)\;-\;\frac{1}{1-2\eta}\;-\;2\,\gamma_{\scriptscriptstyle E}\;-\;2\,\psi(1-2\eta)\;+\;\frac{2\pi\cot(2\pi\eta)}{1-a^2}$$

$$+\;O\big((1-z\bar z)\log(1-z\bar z)\big)\;. \qquad (4.23)$$

We notice from this formula that the two limits $a^2\to 1$ and $|z|\to 1$ of $g(z,z)$ do not commute.

The regularized value of the Green function on the boundary is defined in (3.35). Again, its explicit expression can be obtained either by taking the limit (3.35) on (4.20) or by summing explicitly the series given by

$$g\big(e^{i\theta},e^{i\theta'}\big)\;=\;a_0(1)\,b_0(1)+\sum_{m\geqslant 1}a_m(1)\,b_m(1)\,\cos\big(m(\theta-\theta')\big) \qquad (4.24)$$

and

$$-\sum_{m\geqslant 1}\frac{2}{m}\,\cos\big(m(\theta-\theta')\big)\;=\;\log\big|\,e^{i\theta}-e^{i\theta'}\big|^2\;=\;\log\big(2-2\cos(\theta-\theta')\big)$$

$$=\;2\,\log\big|\theta-\theta'\big|+O\big((\theta-\theta')^2\big)\;. \qquad (4.25)$$

The result is

$$g_{\scriptscriptstyle B}(\theta,\theta)\;=\;-\,\frac{1}{1-2\eta}\;-\;2\,\gamma_{\scriptscriptstyle E}\;-\;2\,\psi(1-2\eta)\;+\;\frac{2\pi\cot(2\pi\eta)}{1-a^2} \qquad (4.26)$$

which is independent of θ by rotational invariance.

We notice that $g_{\scriptscriptstyle B}(\theta,\theta)$ coincides with the finite part of $g(z,z)$ when $|z|\to 1$, as shown in general in section 3.

JHEP06(2006)022

4.3 Fixed area and boundary length expansion

At the semiclassical level, formula (4.3) coming from the boundary conditions (2.15) for the classical field φ_c tells us that $\mu_B < 0$; hence, from (2.2), we have to work at least with fixed boundary length l. The semiclassical value of the action at fixed area A and fixed boundary length l has been computed in [2] and it has been reported in (4.8).

To compute the quantum determinant at fixed area and boundary length, we perform a constrained functional integral by exploiting the results obtained in section 3 for the N point functions. For the one point function, (3.1) becomes

$$\left\langle e^{2(\eta/b)\phi(0)} \right\rangle \equiv U(\eta;\mu,\mu_B) \equiv \int_0^\infty \frac{dl}{l}\, e^{-\mu_B l} \int_0^\infty \frac{dA}{A}\, e^{-\mu A}\, Z(\eta;A,l)\,. \tag{4.27}$$

In order to understand the dependence of $Z(\eta;A,l)$ on its arguments, it is useful to define $\hat{\varphi}_c$ as follows

$$e^{\varphi_c} = \left(\frac{l}{2\pi}\right)^2 \frac{(1-a^2)^2}{\big((z\bar{z})^\eta - a^2(z\bar{z})^{1-\eta}\big)^2} \equiv \left(\frac{l}{2\pi}\right)^2 e^{\hat{\varphi}_c} \tag{4.28}$$

where $\hat{\varphi}_c$ depends only on η and a^2. Using

$$e^{-\frac{1}{4\pi}\oint_{\partial\Delta}\varphi_c\, d\theta} = \frac{2\pi}{l} \tag{4.29}$$

and the definition (4.28) of $\hat{\varphi}_c$, from (3.6) we find to one loop

$$Z(\eta;A,l) = e^{-S_0(\eta;A,l)/b^2}\frac{(2\pi)^4 A}{2\,b^2 l^3}\int \mathcal{D}[\chi]\; e^{-\frac{1}{2}(\chi,D\chi)}\delta\left(\int_\Delta e^{\hat{\varphi}_c}\chi\, d^2z\right)\delta\left(e^{\hat{\varphi}_c(1)/2}\oint_{\partial\Delta}\chi\, d\theta\right). \tag{4.30}$$

Exploiting the relation (4.6), we get the following structure

$$\begin{aligned} Z(\eta;A,l) &= e^{-S_0(\eta;A,l)/b^2}\,\frac{(2\pi)^4 A}{2\,b^2 l^3}\, f(\eta, A/l^2)\big(1+O(b^2)\big) \\ &= e^{-S_0(\eta;A,l)/b^2}\,\frac{(2\pi)^4 A}{2\,b^2 l^3}\, f_1(\eta, a^2)\big(1+O(b^2)\big)\,. \end{aligned} \tag{4.31}$$

After expanding $\chi(z)$ in circular harmonics

$$\chi(z) = \sum_{m\geqslant 0}\chi_m(x)\cos(m\theta) \qquad\qquad x = |z|^2 \tag{4.32}$$

we notice that the constraints involve only the $m = 0$ component of the quantum field $\chi(z)$; hence we are left with the following constrained quadratic path integral to one loop

$$Z(\eta;A,l) = e^{-S_0(\eta;A,l)/b^2}\frac{(2\pi)^2 A}{2\,b^2 l^3}\int \mathcal{D}[\chi] e^{-\frac{1}{2}(\chi,D\chi)}\,\delta\left(\int_0^1 e^{\hat{\varphi}_c}\chi_0(x)\,dx\right)\delta\Big(e^{\hat{\varphi}_c(1)/2}\,\chi_0(1)\Big). \tag{4.33}$$

The integrations over the partial waves with $m \neq 0$ give no problems because the constraints involve only the $m = 0$ sector of the quadratic functional integral (4.33).

JHEP06(2006)022

4.4 The $m = 0$ sector

In this subsection we shall examine the $m = 0$ subspace. In the appendix is proved that the operator D_0, i.e. D acting on the $m = 0$ subspace, has one and only one negative eigenvalue. To simplify the notation, we shall denote by $\zeta(z)$ the field $\chi_0(z)$, by $\zeta_1(z)$ the normalized eigenfunction of D_0 associated to the unique eigenvalue $\mu_1 = (2/\pi)\lambda_1 < 0$ and by $\zeta_\perp(z)$ the component of $\chi_0(z)$ orthogonal to $\zeta_1(z)$.

First we prove that the fixed boundary length constraint is sufficient to make the functional integral (4.33) stable. Exploiting the integral representation of the δ function, the fixed boundary length constrained path integral is given by

$$Y = \frac{1}{2\pi}\int \mathcal{D}[\,\zeta\,]\int d\tau\ \exp\left\{-\frac{1}{2}\left(\zeta, D_0\zeta\right) + i\,\tau\, e^{\hat{\varphi}_c(1)/2}\,\zeta(1)\right\} \tag{4.34}$$
$$= \frac{1}{2\pi}\int \mathcal{D}[\,\zeta_\perp\,]\int d\tau \int dc_1\ \exp\left\{-\frac{\mu_1}{2}\,c_1^2 - \frac{1}{2}\left(\zeta_\perp, D_0\zeta_\perp\right) + i\,\tau\, e^{\hat{\varphi}_c(1)/2}\left(c_1\zeta_1(1) + \zeta_\perp(1)\right)\right\}$$

where $\zeta(z) = c_1\zeta_1(z) + \zeta_\perp(z) = \sum_{n=1}^{+\infty} c_n\zeta_n(z)$. Now we perform the following change of variable

$$\zeta_\perp(z) \ = \ \zeta'_\perp(z) \ + \ i\,\tau\, g_{0\perp}(z,1)\, e^{\hat{\varphi}_c(1)/2} \tag{4.35}$$

where

$$g_{0\perp}(z,z') \ = \ \sum_{n\geqslant 2} \frac{\zeta_n(z)\zeta_n(z')}{\mu_n} \tag{4.36}$$

is the Green function of the $m = 0$ sector orthogonal to the mode $\zeta_1(z)$. Then, integrating in τ, we find

$$Y \ = \ \frac{1}{\sqrt{2\pi\, e^{\hat{\varphi}_c(1)} g_{0\perp}(1,1)}}\int \mathcal{D}[\,\zeta'_\perp\,]\int dc_1\ \exp\left\{-\frac{1}{2}\left(\zeta'_\perp, D_0\zeta'_\perp\right) - \frac{c_1^2}{2}\left(\mu_1 + \frac{\zeta_1^2(1)}{g_{0\perp}(1,1)}\right)\right\} \tag{4.37}$$

where $g_{0\perp}(1,1) > 0$ because $\mu_j > 0$ for $j \geqslant 2$. The coefficient of $-c_1^2/2$ can be written in the following form

$$\frac{\mu_1}{g_{0\perp}(1,1)}\, g_0(1,1) \tag{4.38}$$

from which one immediately sees that it is strictly positive, being

$$g_0(1,1) \ = \ a_0(1)\, b_0(1) \ = \ -\,\frac{1+a^2}{(1-2\eta)(1-a^2)} \ < \ 0\,. \tag{4.39}$$

Now we can integrate in c_1 and the final result for Y is

$$Y \ = \ \frac{1}{\sqrt{-\mu_1}\sqrt{-e^{\hat{\varphi}_c(1)} g_0(1,1)}}\int \mathcal{D}[\,\zeta_\perp\,]\, e^{-\frac{1}{2}(\zeta_\perp, D\zeta_\perp)}\,. \tag{4.40}$$

This procedure shows that in spite of $\mu_1 < 0$ the constrained integral is stable.
Thus one could work keeping fixed μ and the boundary length l. Instead, to compare our results with the ones obtained in [2], we introduce also the fixed area constraint. Exploiting again the integral representation of the δ functions, the functional integral for the $m = 0$

JHEP06(2006)022

wave coming from (4.33) reads

$$\frac{1}{(2\pi)^2}\int\mathcal{D}[\zeta]\int d\rho\int d\tau\ \exp\left\{-\frac{1}{2}\left(\zeta,D_0\zeta\right)+i\,\rho\int_0^1 e^{\hat{\varphi}_c}\zeta(x)\,dx+i\,\tau\,e^{\hat{\varphi}_c(1)/2}\,\zeta(1)\right\}. \tag{4.41}$$

Separating the mode relative to the negative eigenvalue μ_1 and proceeding as shown before, we get the following result for the contribution $Z_0(\eta;A,l)$ of the $m=0$ wave to $Z(\eta;A,l)=e^{-S_0(\eta;A,l)/b^2}\prod_{m=0}^{+\infty}Z_m(\eta;A,l)$ to one loop

$$\begin{aligned}Z_0(\eta;A,l) &= \frac{\pi A}{b^2l^3}\,\frac{1}{(-\det\hat{M}_0)^{1/2}}\,\frac{\sqrt{2\pi}}{\sqrt{-\mu_1}}\int\mathcal{D}[\zeta_\perp]\,e^{-\frac{1}{2}(\zeta_\perp,D_0\zeta_\perp)}\\ &= \frac{\pi A}{b^2l^3}\,\frac{1}{(-\det\hat{M}_0)^{1/2}}\,\frac{1}{(-\mathrm{Det}\,D_0)^{1/2}}\end{aligned} \tag{4.42}$$

where

$$\det\hat{M}_0 = e^{\hat{\varphi}_c(1)}\left[g_0(1,1)\int_0^1\int_0^1 e^{\hat{\varphi}_c(x)}g_0(x,y)\,e^{\hat{\varphi}_c(y)}dx\,dy-\left(\int_0^1 g_0(x,1)\,e^{\hat{\varphi}_c(x)}dx\right)^2\right]. \tag{4.43}$$

Using the explicit expressions for $e^{\hat{\varphi}_c}$ and $g_0(z,z')$, we get

$$\det\hat{M}_0 = -\frac{(1-a^2)^2}{4(1-2\eta)^4} = -\left(\frac{2\pi A}{l^2}\right)^2\frac{1}{(1-2\eta)^2}\,. \tag{4.44}$$

Summing up, our procedure has lead us to the following expression

$$Z(\eta;A,l) = e^{-S_0(\eta;A,l)/b^2}\,\frac{(2\pi)^4A}{2\,b^2l^3}\,f_1(\eta,a^2)\big(1+O(b^2)\big) \tag{4.45}$$

$$= e^{-S_0(\eta;A,l)/b^2}\,\frac{\pi A}{b^2l^3}\,\frac{1}{(-\det\hat{M}_0)^{1/2}}\,\frac{1}{(-\mathrm{Det}\,D)^{1/2}}\big(1+O(b^2)\big) \tag{4.46}$$

$$= e^{-S_0(\eta;A,l)/b^2}\,\frac{1-2\eta}{2b^2l}\,\frac{1}{(-\mathrm{Det}\,D)^{1/2}}\big(1+O(b^2)\big) \tag{4.47}$$

where the remaining quadratic path integral $(-\mathrm{Det}\,D)^{-1/2}$ involves all the waves $m\geqslant 0$

$$(-\mathrm{Det}\,D)^{-1/2} = \frac{\sqrt{2\pi}}{\sqrt{-\mu_1}}\int\mathcal{D}[\chi_\perp]\,e^{-\frac{1}{2}(\chi_\perp,D\chi_\perp)} \tag{4.48}$$

and it is unconstrained.

4.5 The one point function to one loop

The unconstrained functional integral occurring in (4.47) must be computed with the boundary conditions

$$-r^2\frac{\partial}{\partial r^2}\chi(z) = \pi\,\mu_B b^2\,e^{\varphi_c/2}\,\chi(z)\qquad\text{when}\qquad r^2=1\,. \tag{4.49}$$

To determine the function $f_1(\eta,a^2)\equiv f(\eta,A/l^2)$ in (4.45) we shall compute the derivatives of $\log(-\mathrm{Det}\,D)^{-1/2}$ w.r.t. η and a^2 by exploiting (3.26). Indeed, from (4.9) and (4.11) one

JHEP06(2006)022

sees that $(-\mathrm{Det}\,D)^{-1/2}$ depends only on η and a^2. By using the explicit expressions for $g(z,z)$ and $g_B(\theta,\theta)$ in (3.26), given by (4.22) and (4.26) respectively, we find that

$$\frac{\partial}{\partial\eta}\log(-\mathrm{Det}\,D)^{-1/2}\Big|_{a^2} = 2\,\gamma_E + \frac{1}{1-2\eta} + 2\,\psi(1-2\eta) - 2\pi\cot(2\pi\eta) \tag{4.50}$$

$$\frac{\partial}{\partial a^2}\log(-\mathrm{Det}\,D)^{-1/2}\Big|_{\eta} = \frac{1}{1-a^2}\,. \tag{4.51}$$

Combining these results, we obtain

$$(-\mathrm{Det}\,D)^{-1/2} = \frac{\beta}{1-a^2}\,\frac{e^{2\eta\gamma_E}\,\Gamma(2\eta)}{\pi\sqrt{1-2\eta}} \tag{4.52}$$

where β is a numerical factor.

Exploiting the relation (4.6) and the expression (4.47), the one point function at fixed area and boundary length reads

$$Z(\eta;A,l) = e^{-S_0(\eta;A,l)/b^2}\,\frac{\beta}{8\pi^2}\,\frac{l}{b^2A}\,\frac{e^{2\eta\gamma_E}\,\Gamma(2\eta)}{\sqrt{1-2\eta}}\left(1+O(b^2)\right). \tag{4.53}$$

The bootstrap approach gives for the one point function at fixed area and boundary length the following result [2]

$$Z_{\eta/b}(A,l) = \frac{1}{b}\,\frac{\Gamma(2\eta-b^2)}{\Gamma(1+(1-2\eta)/b^2)}\left(\frac{l\,\Gamma(b^2)}{2\,A}\right)^{\frac{1-2\eta}{b^2}+1}\exp\left(-\frac{l^2}{4A\sin(\pi b^2)}\right). \tag{4.54}$$

The one loop expansion of (4.54) is[2]

$$\begin{aligned}Z_{\eta/b}(A,l) = \exp\left\{-\frac{1}{b^2}\left[\frac{l^2}{4\pi\,A}+(1-2\eta)\left(\log\frac{2A}{l}+\log(1-2\eta)-1\right)\right]\right\}\times\\ \times\frac{e^{-\gamma_E}}{2\sqrt{2\pi}}\,\frac{l}{b^2A}\,\frac{e^{2\eta\gamma_E}\,\Gamma(2\eta)}{\sqrt{1-2\eta}}\end{aligned} \tag{4.55}$$

which agrees with (4.53), except for the arbitrary normalization constant β. Eq. (4.53) provides the first perturbative check of the bootstrap result (4.54).
Integrating back (4.53) in A we obtain

$$\begin{aligned}\int_0^\infty\frac{dA}{A}\,e^{-\mu A}\,Z(\eta;A,l) = e^{\frac{1-2\eta}{b^2}\,(1-\log(1-2\eta))}\,(\pi\mu b^2)^{\frac{1-2\eta}{2b^2}+\frac{1}{2}}\times\\ \times\frac{\beta}{2\pi^2b^2}\,\frac{e^{2\eta\gamma_E}\,\Gamma(2\eta)}{\sqrt{1-2\eta}}\,K_{\frac{1-2\eta}{b^2}+1}\left(\sqrt{\frac{\mu}{\pi b^2}}\,l\right)\left(1+O(b^2)\right)\end{aligned} \tag{4.56}$$

and integrating further this result in l according to (4.27) we find to one loop

$$\begin{aligned}U(\eta;\mu,\mu_B) = e^{\frac{1-2\eta}{b^2}\left(\frac{1}{2}\log(\pi\mu b^2)+1-\log(1-2\eta)\right)}\times\\ \times\sqrt{\pi\mu b^2}\,\frac{\beta}{2\pi b^2}\,\frac{e^{2\eta\gamma_E}\,\Gamma(2\eta)}{\sqrt{1-2\eta}}\,\frac{\cosh\left(\pi\sigma\left((1-2\eta)/b^2+1\right)\right)}{\left((1-2\eta)/b^2+1\right)\,\sin\left(\pi(1-2\eta)/b^2\right)}\end{aligned} \tag{4.57}$$

[2] Here we correct a misprint occurring in [2, eq. (2.48)].

JHEP06(2006)022

where σ is defined as follows [2]

$$\big(\cosh(\pi\sigma)\big)^2 \equiv \frac{\mu_B^2}{\mu}\,\pi b^2 \,. \tag{4.58}$$

We notice that the factor $1/\sin\big(\pi(1-2\eta)/b^2\big)$, which displays infinite poles for $b^2 \to 0$, is due to a divergence at the origin in the Laplace transform in l.

The expression (4.57) agrees with the one loop expansion of the bootstrap formula [2, 3]

$$U(\alpha;\mu,\mu_B) = \frac{2}{b}\big(\pi\mu\gamma(b^2)\big)^{\frac{Q-2\alpha}{2b}}\,\Gamma(2\alpha b - b^2)\,\Gamma\left(\frac{2\alpha}{b} - \frac{1}{b^2} - 1\right)\cosh\big(\pi s(2\alpha - Q)\big) \tag{4.59}$$

where $Q = 1/b + b$, $\gamma(x) = \Gamma(x)/\Gamma(1-x)$ and the parameter s is defined by

$$\big(\cosh(\pi b s)\big)^2 = \frac{\mu_B^2}{\mu}\,\sin(\pi b^2)\,. \tag{4.60}$$

We notice that in the limit $a^2 \to 1$ the semiclassical contribution to $U(\eta;\mu,\mu_B)$ in (4.57), which is

$$e^{\frac{1-2\eta}{b^2}\left(\frac{1}{2}\log(\pi\mu b^2)+1-\log(1-2\eta)-\frac{1}{2}\log a^2\right)} = e^{-S_{cl}[\varphi_c]} \tag{4.61}$$

goes over to the semiclassical result of the pseudosphere [1], up to an η independent normalization constant. On the other hand the quantum contribution develops an infinite number of poles for $b \to 0$, as discussed after (4.58).

In principle the method can be extended to higher loop even if it appears computationally rather heavy.

5. Conclusions

The extension of the technique developed in [1] for the pseudosphere has been successfully applied to the conformal boundary case.

A general method has been found for treating functional integrals with constraints, like the fixed area and boundary length constraints. We proved that, by properly regularizing the Green function, the correct quantum dimensions for the vertex functions are recovered. We gave the explicit computation of the one point function at fixed area and boundary length to one loop, providing the first perturbative check of the results obtained through the bootstrap method [2, 3].

Acknowledgments

We are grateful to Damiano Anselmi, Giovanni Morchio and Massimo Porrati for useful discussions.

JHEP06(2006)022

A. The spectrum of the D operator

Here we examine the spectrum of the operator

$$\Theta \equiv \frac{\pi}{2} D = -\partial_z\partial_{\bar z} + 2\pi\mu b^2 e^{\varphi_c} = -\partial_z\partial_{\bar z} + \frac{2a^2(1-2\eta)^2}{\big((z\bar z)^\eta - a^2(z\bar z)^{1-\eta}\big)^2} \tag{A.1}$$

with boundary conditions (4.11)

$$\big(z\,\partial_z + \bar z\,\partial_{\bar z}\big)\,\chi(z) = (1-2\eta)\frac{1+a^2}{1-a^2}\,\chi(z) \qquad \text{when} \qquad |z| = 1 \tag{A.2}$$

where e^{φ_c} is given in (4.1).

Considering the wave $m = 0$, the eigenvalue equation with eigenvalue λ

$$\Theta_0\,\chi = \frac{\pi}{2} D_0\,\chi = \lambda\,\chi \tag{A.3}$$

can be rewritten as

$$-(y\,\chi')' + \frac{2}{(1-y)^2}\,\chi = y^\rho \Lambda\,\chi \tag{A.4}$$

where $y = a^2(z\bar z)^{1-2\eta}$, $\rho = 2\eta/(1-2\eta)$ and

$$\Lambda = \frac{\lambda}{(1-2\eta)^2(a^2)^{1/(1-2\eta)}}\;. \tag{A.5}$$

The boundary conditions (A.2) read

$$\frac{\chi'}{\chi}\bigg|_{y=a^2} = \frac{1+a^2}{2a^2(1-a^2)} \tag{A.6}$$

and $\chi(y)$ is regular at the origin. For $\Lambda = 0$, the solution of (A.4) which is regular at the origin is

$$f_0 = \frac{1+y}{1-y} \tag{A.7}$$

i.e. the function a_0 given in (4.17), but it does not satisfy the boundary conditions (A.6) because

$$\frac{f_0'}{f_0}\bigg|_{y=a^2} = \frac{2}{1-a^4} < \frac{1+a^2}{2a^2(1-a^2)} \tag{A.8}$$

being $a^2 < 1$. Thus we have

$$\begin{aligned} 0 = \int_0^{a^2} f_0(\Theta f_0)\,dy &= \int_0^{a^2}\left(y(f_0')^2 + \frac{2f_0^2}{(1-y)^2}\right)dy - a^2 f_0'(a^2)f_0(a^2) \\ &= \int_0^{a^2}\left(y(f_0')^2 + \frac{2f_0^2}{(1-y)^2}\right)dy - \frac{2a^2(1+a^2)}{(1-a^2)^3} \end{aligned} \tag{A.9}$$

i.e.

$$\int_0^{a^2}\left(y(f_0')^2 + \frac{2f_0^2}{(1-y)^2}\right)dy = \frac{2a^2(1+a^2)}{(1-a^2)^3}\;. \tag{A.10}$$

JHEP06(2006)022

Now it is easy to modify slightly f_0 near $y = a^2$ to a function f_ε satisfying the boundary conditions (A.6) and for which

$$\int_0^{a^2} f_\varepsilon(\Theta f_\varepsilon)\, dy = \int_0^{a^2} \left(y(f_\varepsilon')^2 + \frac{2f_\varepsilon^2}{(1-y)^2} \right) dy - a^2 f_\varepsilon'(a^2) f_\varepsilon(a^2) \tag{A.11}$$

with

$$\lim_{\varepsilon\to 0} f_\varepsilon'(a^2) f_\varepsilon(a^2) = \frac{(1+a^2)^3}{2a^2(1-a^2)^3} \tag{A.12}$$

and

$$\lim_{\varepsilon\to 0} \int_0^{a^2} \left(y(f_\varepsilon')^2 + \frac{2f_\varepsilon^2}{(1-y)^2} \right) dy = \int_0^{a^2} \left(y(f_0') + \frac{2f_0^2}{(1-y)^2} \right) dy = \frac{2a^2(1+a^2)}{(1-a^2)^3} \; . \tag{A.13}$$

Being $a^2 < 1$, we have that

$$\frac{2a^2(1+a^2)}{(1-a^2)^3} < \frac{(1+a^2)^3}{2(1-a^2)^3} \tag{A.14}$$

and therefore on such test function f_ε, which is not an eigenfunction, we have

$$\int_0^{a^2} f_\varepsilon \, \Theta f_\varepsilon \, dy \; < \; 0 \tag{A.15}$$

for sufficiently small ε. This proves that the operator Θ is not positive definite, i.e. it possesses at least one negative eigenvalue $\lambda_1 < 0$.

We want now to prove that the ground eigenvalue λ_1 is the only negative eigenvalue occurring in the spectrum. First we write the eigenvalue equation (A.4) as

$$(y\chi')' = \left(\frac{2}{(1-y)^2} \; - \; y^\rho \Lambda \right) \chi \; . \tag{A.16}$$

The solution of (A.16) which is regular at the origin can be written as the following convergent series

$$\chi = \chi^{(0)} + \chi^{(1)} + \chi^{(2)} + \ldots \tag{A.17}$$

with $\chi^{(0)} = 1$ and

$$\chi^{(n)} = \int_0^y (\log y - \log y_1) \left(\frac{2}{(1-y_1)^2} \; - \; y^\rho \Lambda \right) \chi^{(n-1)}(y_1)\, dy_1 \; . \tag{A.18}$$

From (A.17) and (A.18), one immediately realizes that for $\Lambda < 0$ the function χ is a positive function, increasing in y and a pointwise increasing function of $-\Lambda$. Since $\Lambda_1 < 0$, the ground state eigenfunction is a positive function. The eigenfunction relative to $\Lambda_2 > \Lambda_1$ must possess, by orthogonality, at least one node, but, as we cannot have a node for $\Lambda_2 \leqslant 0$, we must have $\Lambda_2 > 0$. Thus the operator Θ with boundary conditions (A.2) has one and only one negative eigenvalue. The presence of a negative eigenvalue makes the unconstrained functional integral ill defined.

JHEP06(2006)022

Obviously one has to consider also the positivity of the partial wave operator for $m = 1$ and higher m. The eigenvalue equation in $y = a^2u = a^2(z\bar{z})^{1-2\eta}$ for $m \geqslant 1$ is

$$-\,(y\chi')' + \frac{m^2}{4(1-2\eta)^2}\,\frac{\chi}{y} + \frac{2}{(1-y)^2}\,\chi \,=\, y^\rho \Lambda\,\chi\,. \tag{A.19}$$

It will be sufficient to examine the case $m = 1$. The iterative solution of the following equation

$$(y\chi')' - \frac{1}{4(1-2\eta)^2}\,\frac{\chi}{y} \,=\, \left(\frac{2}{(1-y)^2} - y^\rho\Lambda\right)\chi \tag{A.20}$$

is provided by series (A.17) with

$$\chi^{(0)} = y^{\gamma/2} \tag{A.21}$$

$$\chi^{(n)} = \frac{1}{\gamma}\int_0^y (y^{\gamma/2}y_1^{-\gamma/2} - y^{-\gamma/2}y_1^{\gamma/2})\left(\frac{2}{(1-y_1)^2} - y^\rho\Lambda\right)\chi^{(n-1)}(y_1)\,dy_1 \tag{A.22}$$

where $\gamma = 1/(1-2\eta)$. Since we have always $y_1 \leqslant y$, then

$$y^{\gamma/2}y_1^{-\gamma/2} - y^{-\gamma/2}y_1^{\gamma/2} \,\geqslant\, 0\,. \tag{A.23}$$

Again, being $\chi^{(0)} > 0$, we have that the terms of the series for $\Lambda \leqslant 0$ are positive increasing in y and pointwise increasing in $-\Lambda$. For $m = 1$ and $\eta = 0$ we know a solution of the equation with null eigenvalue. It is

$$\chi = \frac{y^{\frac{1}{2}}}{1-y} \tag{A.24}$$

which gives

$$\frac{\chi'}{\chi} = \frac{1+y}{2y(1-y)} \tag{A.25}$$

i.e. it satisfies identically the boundary conditions (A.6). Thus for $m = 1$ and $\eta = 0$ we have the marginal eigenvalue $\Lambda = 0$. Since χ pointwise increases when $-\Lambda$ increases, then we cannot have nodes for $\Lambda < 0$ and, by orthogonality, we cannot have eigenvalues for $\Lambda < 0$ either. Thus, for $m = 1$ and $\eta = 0$ the operator is positive semidefinite. Then, from (A.19), we see that the operator is positive definite when $m > 1$ and $\eta \geqslant 0$ (always $\eta < 1/2$). For $m = 1$ and $\eta < 0$ the operator is not positive definite (use as test function the solution (A.24) for $m = 1$ and $\eta = 0$) and therefore, when $\eta < 0$, we have instability also for the $m = 1$ wave.

References

[1] P. Menotti and E. Tonni, *Liouville field theory with heavy charges, I. The pseudosphere*, hep-th/0602206.

[2] V. Fateev, A.B. Zamolodchikov and A.B. Zamolodchikov, *Boundary Liouville field theory, I. boundary state and boundary two-point function*, hep-th/0001012.

[3] J. Teschner, *Remarks on Liouville theory with boundary*, hep-th/0009138.

[4] B. Ponsot and J. Teschner, *Boundary Liouville field theory: boundary three point function*, *Nucl. Phys.* **B 622** (2002) 309 [hep-th/0110244].

JHEP06(2006)022

[5] K. Hosomichi, *Bulk-boundary propagator in Liouville theory on a disc*, *JHEP* **11** (2001) 044 [hep-th/0108093].

[6] A.B. Zamolodchikov and A.B. Zamolodchikov, *Liouville field theory on a pseudosphere*, hep-th/0101152.

[7] P. Menotti and E. Tonni, *Standard and geometric approaches to quantum Liouville theory on the pseudosphere*, *Nucl. Phys.* **B 707** (2005) 321 [hep-th/0406014].

[8] P. Menotti and E. Tonni, *The tetrahedron graph in Liouville theory on the pseudosphere*, *Phys. Lett.* **B 586** (2004) 425 [hep-th/0311234].

[9] J. Teschner, *Liouville theory revisited*, *Class. and Quant. Grav.* **18** (2001) R153 [hep-th/0104158].

[10] T.L. Curtright and C.B. Thorn, *Conformally invariant quantization of the Liouville theory*, *Phys. Rev. Lett.* **48** (1982) 1309.

[11] A.B. Zamolodchikov and A.B. Zamolodchikov, *Structure constants and conformal bootstrap in Liouville field theory*, *Nucl. Phys.* **B 477** (1996) 577 [hep-th/9506136].

[12] N. Seiberg, *Notes on quantum Liouville theory and quantum gravity*, *Prog. Theor. Phys. Suppl.* **102** (1990) 319.

[13] E. Picard, *De l'equation $\Delta u = ke^u$ sur une surface de Riemann fermee*, *J. Math. Pures Appl.* 4 (1893) 273.

[14] P.H. Ginsparg and G.W. Moore, *Lectures on 2D gravity and 2D string theory*, hep-th/9304011.

[15] J. Ambjørn, K. Hayasaka and R. Nakayama, *Non-critical open strings beyond the semi-classical approximation*, *Mod. Phys. Lett.* **A 12** (1997) 1241 [hep-th/9702019]; P. Mansfield and R. Neves, *Boundary conformal field theories on random surfaces and the non-critical open string*, *Nucl. Phys.* **B 479** (1996) 82 [hep-th/9605097].

[16] P.R. Garabedian, *Partial differential equations*, John Wiley & Sons, New York, 1964.

[17] G. Moore, N. Seiberg and M. Staudacher, *From loops to states in two-dimensional quantum gravity*, *Nucl. Phys.* **B 362** (1991) 665.

[18] A. Erdelyi, *Higher transcendental functions*, vol. I, McGraw-Hill, New York, 1953; A. Prudnikov, Y. Brychkov and I. Marichev, *Integral and series*, vol. III, Gordon and Breach, New York, 1986.

[19] P. Menotti and E. Tonni, *Quantum Liouville theory on the pseudosphere with heavy charges*, *Phys. Lett.* **B 633** (2006) 404 [hep-th/0508240].

[20] P. Menotti and G. Vajente, *Semiclassical and quantum Liouville theory on the sphere*, *Nucl. Phys.* **B 709** (2005) 465 [hep-th/0411003].

JHEP06(2006)022